

Motion around a monopole + ring system – I. Stability of equatorial circular orbits versus regularity of three-dimensional motion

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ABSTRACT

We study the motion of test particles around a centre of attraction represented by a monopole (with and without spheroidal deformation) surrounded by a ring, given as a superposition of Morgan and Morgan discs. We deal with two kinds of bounded orbits: (i) equatorial circular orbits and (ii) general three-dimensional orbits. The first case provides a method to perform a linear stability analysis of these structures by studying the behaviour of vertical and epicyclic frequencies as functions of the mass ratio, the size of the ring and/or the quadrupolar deformation. In the second case, we study the influence of these parameters in the regularity or chaoticity of motion. We find that there is a close connection between linear stability (or instability) of equatorial circular orbits and regularity (or chaoticity) of the three-dimensional motion.

Key words: chaos – celestial mechanics – planets and satellites: rings.

1 INTRODUCTION

Light particles interacting with a central massive body are frequently encountered in diverse fields of physics. Electrons in rotating molecules and particular versions of the many-body problem in celestial mechanics are the most well-known examples, but some nuclear models fall into the same category. Increasing amounts of information about narrow planetary rings suggest that such rings are often associated to the so-called shepherd satellites (Goldreich & Tremaine 1979), and may exist due to mechanisms somewhat more complicated than the well-known broad rings (Smith et al. 1981, 1982; Greenberg & Brahic 1984; Murray et al. 2005).

Recent progress in the subject suggests a generic mechanism that does not depend on Kepler orbits to explain the formation of rings around a rotating object which also holds for systems quite different from planetary rings. In rotating scattering systems, the generic saddle-centre scenario leads to stable islands in phase space. Non-interacting particles whose initial conditions are defined in such islands will be trapped and form rotating rings. This result is generic and guarantees that the orbits supporting the ring structure are rather insensitive to small perturbations and thus may play a role in different situations of the type mentioned above (Benet & Seligman 2000). Therefore, although we are interested in studying planetary rings, there are many other systems that exhibit the same structure and therefore one can perform similar treatments by knowing the nature of the forces involved.

Now, based on the arguments mentioned above, one may conjecture that once the ring structure is formed around the rotating body, it remains stable, as revealed by the structure of its phase space. In this set of papers, we propose an analytical study about the dynamical aspects concerning the stability of ring structures and the relation with the existence of isolated islands in the phase space. In this first paper we focus on the effects related to the physical parameters like the mass of the central body, the geometry of the central body, the mass of the ring and the size of the ring. In next papers, we plan to examine the influence of the angular momentum of the system in its stability as well as large perturbations like the interaction with external satellites and their responses to resonance.

Planetary rings consist in thin discs of cosmic dust and other small colliding particles revolving around a central planet in a flat disc-shaped region. The most spectacular examples of ring structures are those around Saturn (Porco et al. 2005; Cuzzi et al. 2010), but they are a common feature of the other three gas giants of the Solar system; Jupiter, Uranus and Neptune possess ring systems of their own. Recent reports have suggested that the Saturnian moon Rhea may have its own tenuous ring system, which would make it the only moon known to possess a ring system (Jones et al. 2008).

There are many possible mechanisms to explain the existence of planetary rings, but essentially three of them are the most relevant: from material of the protoplanetary disc that was within the Roche limit of the planet and thus could not coalesce to form moons; from the debris of a moon that was disrupted by a large impact or from the debris of a moon that was disrupted by tidal stresses when it passed within the planet's Roche limit. This last one allows us to predict that Phobos, a moon of Mars, will break up and form into a planetary ring in about 50 Myr due to its low orbit (Holsapple 2001).

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Additionally, we should take into account the increasing amount of data we get from extrasolar systems. The discovery of extrasolar planets (which to date are more than 300) by radial-velocity measurements has provided the first dynamical characteristics of planets: orbital elements and mass. The next step will be to investigate physical characteristics: albedo, temperature, radius, etc., and their surroundings. Among the latter are planetary rings. The emitted thermal infrared light from the planet should show no phase effect assuming that the planet is in thermal equilibrium. But the reflected visible light will vary with phase angle, as should be shown by a broad-band photometric follow-up of the planet during its orbital motion. In particular, it has been studied from different perspectives how the presence of a ring around a planet would influence its brightness as a function of its orbital position and based on that there have been multiple theoretical predictions for photometric and spectroscopic signatures of rings around transiting extrasolar planets (Barnes & Fortney 2004; Arnold & Schneider 2006; Ohta, Taruya & Suto 2009). Therefore, understanding the dynamics of composed planetary systems would not constitute just an astrophysical curiosity, it would have some impact on the strategy for their detection (see Arnold & Schneider 2004, and the references therein).

On the other hand, it has been shown that collisions play an important role in the dynamical aspects of the rings (Longaretti 1992; Sicardy 2006). Under some physical assumptions, one can see that as this process dissipates mechanical energy, while conserving angular momentum of the ensemble, tend to flatten the disc perpendicular to its total angular momentum, and to circularize the orbits of the particles. If the planet is not spherically symmetric, as it is the case for all the oblate giant planets, apart from being flat the configuration becomes an equatorial ring. Of course, some physics is still missing in the description of the rings and an interesting question here would be if these effects that one does not take into account as a first approximation are relevant in the dynamics of the ring. A classical approach is to consider what happens to the system when small perturbations are applied.

Commonly, disc-like systems such as planetary rings are described by a set of coupled equations: two hydrodynamic equations (Euler's and continuity equation), Poisson's equation and a equation of state (Toomre 1964; Binney & Tremaine 2008). For small enough disturbances, these equations can be linearized and solved under further simplifications as there are still missing analytical self-consistent models of planetary rings. Nevertheless, it is possible to obtain useful information of this by using simplified models which illustrates quantitatively how pressure and rotation tend to stabilize the disc against self-gravity.

Although most rings were thought to be unstable and to dissipate over the course of tens or hundreds of millions of years, recent evidence coming from NASA's Cassini-Huygens Mission suggest that Saturn's rings might be quite old, dating to the early days of the Solar system. Then, the exact mechanism to explain which physical factors account for the stability of these systems is still an open question to be answered.

Our approach to study the stability of rings will be quite different from that mentioned above. The method we will use is insensitive to the hydrodynamical aspects of the rings but with the advantage that we will have under control the geometrical aspects of the models, namely the size and shape of the ring, the mass quotient between the planet and the ring and deviations from the spherical geometry of the planet. Moreover, similar techniques can be applied to study the effects caused by the angular momentum of the planet and even the influence of external satellites, but these topics are out of scope of

this paper and will be left for future works. Another benefit of this approach is that it becomes easy to elucidate a non-trivial connection between the stability in equatorial circular orbits and the regularity in three-dimensional motion. In particular, we will see how the existence of isolated islands will lead to stable ring configurations and how relevant are the physical parameters mentioned above.

Now, the exact gravitational potential of a ring of zero thickness and constant linear density is given in its exact form by an elliptic integral that is seldom used for practical purposes. This potential is usually approximated by a truncated series of spherical harmonics, i.e. a multipolar expansion (see for example the deep analysis of orbits performed in Tresaco & Ferrer 2010). As far as we know there are no simple expressions for the exact gravitational potential of a finite flat ring with inner and outer edges and any surface density.

Meanwhile, simple potential-density pairs for thin discs are known. The simpler is the Plummer-Kuzmin disc (Kuzmin 1956) that represents a simple model with a concentration of mass in its centre and density that decays as $1/r^3$ on the plane of the disc. This structure has no boundary even though for practical purposes one can put a cut-off radius wherein the main part of the mass is inside, say 98 per cent of the mass. There are many other models in the literature (Lynden-Bell 1962; Mestel 1963; Toomre 1963; Hunter & Toomre 1969; Kalnajs 1972; Jiang 2000; Jiang & Moss 2002; González & Reina 2006; Jiang & Ossipkov 2007; Pedraza, Ramos-Caro & González 2008), some of them of infinite extension but others with an outer edge. Among the last ones, a family of simple models are the Morgan and Morgan discs (Morgan & Morgan 1969), which has been inverted (Lemos & Letelier 1994) in order to obtain infinite discs with a central hole of the same radius of the original disc. We can also put a cut-off in these structures and therefore the inverted Morgan and Morgan discs can be considered as representing flat rings. These models have been used by the authors in order to study the superposition of an annular disc with a central black hole in the context of the General Theory of Relativity (see also Semerák & Suková 2010; Lora-Clavijo, Ospina-Henao & Pedraza 2010, for recent related works).

Another approach to construct flat-ring structures is by the superposition of different kinds of discs. In Letelier (2007) the author superposed Morgan and Morgan discs (Morgan & Morgan 1969), whereas in Vogt & Letelier (2009) the Kuzmin-Toomre discs (Kuzmin 1956; Toomre 1963) were used. In this paper we will use the first of the above models which are relatively simple and superposing the potential of the planet will allow us to study the effects of those parameters we have been talking about.

2 MODELS OF RINGS AROUND A SPHERICALLY SYMMETRIC OBJECT

In this section we shall focus on simple analytical models that represent sources conformed by an axisymmetric thin ring and a central object that we will assume with spherical symmetry, i.e. a body characterized only by its monopolar moment. The effects concerning with oblate or prolate deformation of the centre of attraction, i.e. the inclusion of a quadrupolar moment, will be studied in the last section.

In a recent work, Letelier (2007) shows how to construct a simple family of potential-density pairs for flat rings by means of the superposition of Morgan and Morgan discs (Morgan & Morgan 1969). These discs are finite in extension and have a well-behaved surface mass density with a maximum in the centre and decreasing monotonically up to the edge. The structures obtained by the superposition of discs with different densities have a finite outer

radius and zero density on their centres, i.e. discs with a hole in their centres, or in other words flat rings. Although the models do not have an inner edge, for practical purposes one can put a cut-off radius and neglect the residual density (which becomes smaller for higher members of the family).

The Morgan and Morgan discs are obtained by solving the Laplace equation in the natural coordinates to represent the gravitational potential of a disc-like structure, i.e. oblate coordinates (ξ, η, φ) (defined in the ranges $0 \leq \xi < \infty$ and $-1 \leq \eta \leq 1$) that are related to the usual cylindrical coordinates (R, z, φ) by

$$R = a\sqrt{(\xi^2 + 1)(1 - \eta^2)}, \quad (1)$$

$$z = a\xi\eta,$$

where a is a positive constant defining the disc radius. The inverse relations are given by

$$a\xi = \text{Re} \left[\sqrt{R^2 + (z - ia)^2} \right],$$

$$a\eta = -\text{Im} \left[\sqrt{R^2 + (z - ia)^2} \right]. \quad (2)$$

Note that, according to (1), on the equatorial plane $z = 0$ one has to distinguish between two regions: (i) the points inside the disc, with coordinates $\xi = 0$ and $\eta \leq \sqrt{1 - R^2/a^2}$; (ii) the points outside the disc, with coordinates $\eta = 0$ and $\xi > \sqrt{R^2/a^2 - 1}$.

The mass surface density of each disc (labelled with the positive integer n) is given by

$$\Sigma_n(R) = \frac{(2n + 1)M_n}{2\pi a^2} \left(1 - \frac{R^2}{a^2} \right)^{n-1/2}, \quad (3)$$

where M_n is the total mass and a is the disc radius. Such mass distribution generates an axially symmetric gravitational potential, which can be written as

$$\hat{\Phi}_n(\xi, \eta) = - \sum_{k=0}^n A_{2k,2n} q_{2k}(\xi) P_{2k}(\eta). \quad (4)$$

Here, $P_{2k}(\eta)$ and $q_{2k}(\xi) = i^{2k+1} Q_{2k}(i\xi)$ are the usual Legendre polynomials and the Legendre functions of the second kind, respectively, and $A_{2k,2n}$ are constants given by

$$A_{2k,2n} = \frac{M_n G \pi^{1/2} (4k + 1)(2n + 1)!}{a^{2n+1} (2k + 1)(n - k)! \Gamma(n + k + 3/2) q_{2k+1}(0)}, \quad (5)$$

where G is the gravitational constant. If we consider discs of the same radius a and decreasing mass

$$M_n = \frac{2\pi \Sigma_c a^2}{2n + 1}, \quad (6)$$

where Σ_c is a constant that will be taken equal for all discs of the Morgan and Morgan family, we obtain discs with surface density

$$\Sigma'_n = \Sigma_c \left(1 - \frac{R^2}{a^2} \right)^{1/2} \left(1 - \frac{R^2}{a^2} \right)^{n-1}. \quad (7)$$

In order to obtain a mass density in agreement with a flat-ring distribution, Letelier (2007) considered the superposition

$$\tilde{\Sigma}_n = \sum_{k=0}^n \frac{n!(-1)^{n-k}}{(n-k)!k!} \Sigma'_{n+1-k}$$

$$= \Sigma_c \left(1 - \frac{R^2}{a^2} \right)^{1/2} \frac{R^{2n}}{a^{2n}}. \quad (8)$$

We have that the above superpositions give rings of radius a and a central residual density that becomes smaller for larger n . For practical purposes one can put a cut-off inner radius b_n , which we

Table 1. Values of the ratio b_n/a for the first 10 members of the family of rings given by the mass surface (8). Here b_n represents the inner cut-off radius and a is the outer radius. For higher members of the family the mass concentration tends to be located near the outer edge.

n	b_n/a	n	b_n/a
1	0.062 10	6	0.592 22
2	0.232 92	7	0.643 35
3	0.369 91	8	0.678 84
4	0.439 40	9	0.707 98
5	0.543 02	10	0.732 31

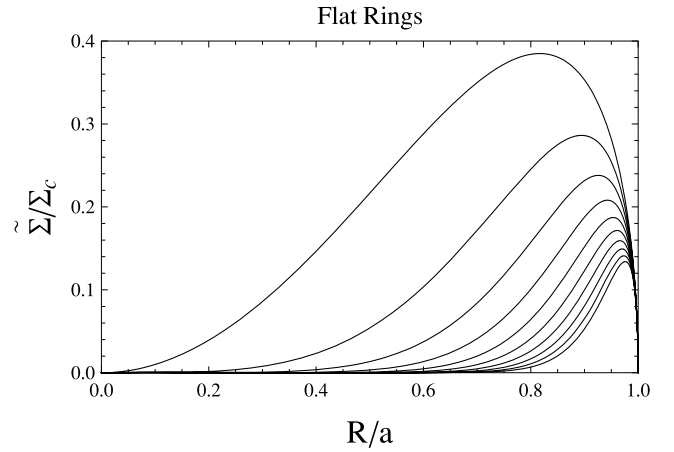


Figure 1. The mass surface density for the first 10 members of the ring family constructed as a superposition of Morgan and Morgan discs.

will take as the radius such that the density falls below 1 per cent of its maximum. In Table 1 we show the values of b_n/a for the first 10 members of this family and, in Fig. 1, their corresponding mass surface densities. The total mass of each ring is

$$\tilde{M}_n = \frac{2\pi \Sigma_c}{a^{2n}} \int_0^a \left(1 - \frac{R^2}{a^2} \right)^{1/2} R^{2n+1} dR$$

$$= \frac{\pi^{3/2} a^2 \Sigma_c \Gamma(n + 1)}{2\Gamma(n + 5/2)}. \quad (9)$$

Thus by increasing n , the mass of the flat ring decreases along with the size of the region where it is distributed (near the outer radius). Note that this is a feature associated to the particular family of models we are considering, not to the corresponding physical applications. The values of n and Σ_c are inferred from the basic data corresponding to a special case. For example, to describe a ring with mass 6×10^{18} kg, inner radius 122 000 km and outer radius 137 000 km, we have to set $n = 28$ and $\Sigma_c \approx 1800 \text{ g cm}^{-2}$, leading to a maximum mass density of about 40 g cm^{-2} (these values agree roughly with the measurements of the so-called Ring A of Saturn; Dougherty, Esposito & Krimigis 2009).

The potentials associated to these structures can be obtained using a superposition with the same coefficients as the ones used for the densities, i.e.

$$\tilde{\Phi}_n = \sum_{k=0}^n \frac{n!(-1)^{n-k}}{(n-k)!k!} \Phi'_{n+1-k}, \quad (10)$$

where Φ'_n is the same as $\hat{\Phi}_n$ with the masses given by equation (6).

Finally we add a monopolar term, which represents the exterior field of a central spherically symmetric object, to the ring potential described above. Thus, the total gravitational potential reads

$$\Phi_n = \tilde{\Phi}_n - \frac{GM_p}{\sqrt{R^2 + z^2}}, \quad (11)$$

where M_p is the mass of the central object. Although our four-parametric toy model is quite simple, we believe that this is a good starting point to study the effects on the dynamics caused by the mass ratio M_n/M_p and the radius ratio a/b_n which is one of the purposes of this paper. The effects caused by the rotation of the central body (its own angular momentum would be an additional parameter) will be studied in a next paper through the post-Newtonian scheme.

3 MOTION OF TEST PARTICLES

Let us deal with the problem of motion of test particles around the models described above. Since Φ_n is static and axially symmetric, the specific energy E and the specific axial angular momentum ℓ are conserved along the particle motion, which is restricted to a three-dimensional subspace of the (R, z, V_R, V_z) phase space. Each orbit is determined by the set of equations (Binney & Tremaine 2008)

$$\dot{R} = V_R, \quad \dot{z} = V_z, \quad (12a)$$

$$\dot{V}_R = -\frac{\partial \Phi_n^*}{\partial R}, \quad \dot{V}_z = -\frac{\partial \Phi_n^*}{\partial z}, \quad (12b)$$

where Φ_n^* is the effective potential, defined by

$$\Phi_n^* = \Phi_n + \frac{\ell^2}{2R^2}, \quad (13)$$

in terms of which the particle's energy can be written as

$$E = \frac{1}{2} (V_R^2 + V_z^2) + \Phi_n^*. \quad (14)$$

Relations (12a)–(12b) define an autonomous system whose equilibrium points are $V_R = V_z = z = 0$ and $R = R_c$, where R_c must satisfy the equation

$$\left(\frac{\partial \Phi_n^*}{\partial R} \right)_{(R_c, 0)} = -\frac{\ell^2}{R_c^3} + \left(\frac{\partial \Phi_n}{\partial R} \right)_{(R_c, 0)} = 0, \quad (15)$$

which is the condition for a circular orbit in the plane $z = 0$. In other words, the equilibrium points of the system occur when the test particle describes equatorial circular orbits of radius R_c , specific energy $E = \Phi_n^*(R_c, 0)$ and specific axial angular momentum given by

$$\ell_c^2 = R_c^3 \left(\frac{\partial \Phi_n}{\partial R} \right)_{(R_c, 0)}. \quad (16)$$

The description of circular orbits in the equatorial plane is a first step to understand the linear stability of the system as well as the regularity or chaoticity of three-dimensional orbits. On one hand, if one assumes as a first approximation that the structures are built from particles moving only in circular orbits, the epicyclic and vertical frequencies of quasi-circular orbits provide us a criterion for the system's stability (see Section 4). On the other hand, the analysis of such frequencies also leads to determine the existence of saddle points, which are preliminary indicators of irregular motion.

In order to deal with the problem of correlation between regularity of three-dimensional motion and the stability of circular orbits, we have to distinguish between exterior and interior motions of test particles, separately. As we pointed out in the last section, the relation between oblate and cylindrical coordinates is given by (2) or in a more explicit form

$$\begin{aligned} \xi &= \Delta(R, z) \cos[\Theta(R, z)], \\ \eta &= \Delta(R, z) \sin[\Theta(R, z)], \end{aligned} \quad (17)$$

where

$$\Delta(R, z) = \left[\frac{4z^2}{a^2} + \left(\frac{R^2 + z^2}{a^2} - 1 \right)^2 \right]^{1/4}, \quad (18)$$

$$\Theta(R, z) = \frac{1}{2} \arctan \left(\frac{2za}{R^2 + z^2 - a^2} \right) \quad (19)$$

for points located outside the disc zone, whereas for points inside the disc region, we have

$$\begin{aligned} \xi &= 0, \\ \eta &= \pm \sqrt{1 - R^2/a^2}, \quad \text{for } z = 0 \text{ and } 0 \leq R \leq a. \end{aligned} \quad (20)$$

The ambiguity in the sign in the last equation is due to the singular behaviour of the coordinate η on crossing the disc. Rigorously speaking we have $\eta = \sqrt{1 - R^2/a^2}$ at $z = 0^+$ (upper limit) and $\eta = -\sqrt{1 - R^2/a^2}$ at $z = 0^-$ (lower limit).

For later formulae, it is important to point out that the piecewise form of this transformation rule leads also to a piecewise form in the first and second derivatives, when evaluated at the equatorial plane. After some calculations, we obtain the following relations for the first derivatives of ξ and η , at $z = 0$:

$$\begin{aligned} \partial \xi / \partial z &= \pm (a^2 - R^2)^{-1/2}, \quad \partial \xi / \partial R = \partial \eta / \partial z = 0, \\ \partial \eta / \partial R &= \mp (R/a)(a^2 - R^2)^{-1/2}, \quad \text{for } 0 \leq R \leq a \end{aligned} \quad (21)$$

and

$$\begin{aligned} \partial \eta / \partial z &= (R^2 - a^2)^{-1/2}, \quad \partial \eta / \partial R = \partial \xi / \partial z = 0, \\ \partial \xi / \partial R &= (R/a)(R^2 - a^2)^{-1/2}, \quad \text{for } R > a. \end{aligned} \quad (22)$$

The ambiguity in the sign in (21) has the same meaning as in (17). These relations are used to compute the second derivatives in the equatorial plane and the result is

$$\begin{aligned} \partial^2 \eta / \partial R^2 &= \mp a(a^2 - R^2)^{-3/2}, \quad \partial^2 \xi / \partial R^2 = \partial^2 \xi / \partial z^2 = 0, \\ \partial^2 \eta / \partial z^2 &= \pm (R^2/a)(a^2 - R^2)^{-3/2}, \quad \text{for } 0 \leq R \leq a \end{aligned} \quad (23)$$

and

$$\begin{aligned} \partial^2 \xi / \partial R^2 &= -a(R^2 - a^2)^{-3/2}, \quad \partial^2 \eta / \partial R^2 = \partial^2 \eta / \partial z^2 = 0, \\ \partial^2 \xi / \partial z^2 &= (R^2/a)(R^2 - a^2)^{-3/2}, \quad \text{for } R > a. \end{aligned} \quad (24)$$

The latter two equations will be important in the calculation of epicyclic and vertical frequencies (see Section 4).

By introducing (21) in the equations of motion (12b), when evaluated in the inner zone, we can derive the equations of motion for a test particle in the case in which $z = 0$, $R \leq a$. For \dot{V}_R we have

$$\dot{V}_R = \frac{\ell^2}{R^3} - \frac{GM_p}{R^2} + \frac{R}{a\sqrt{a^2 - R^2}} \frac{\partial \tilde{\Phi}_n}{\partial \eta} \bigg|_{\eta=\sqrt{1-R^2/a^2}}^{\xi=0}. \quad (25)$$

Note that, in the derivation of this equation, there is no difference between the choice $\eta = \sqrt{1 - R^2/a^2}$ along with $\partial \eta / \partial R = (R/a)(a^2 - R^2)^{-1/2}$, and the other option, $\eta = -\sqrt{1 - R^2/a^2}$ along with $\partial \eta / \partial R = -(R/a)(a^2 - R^2)^{-1/2}$. This is due to the fact that

$\partial\tilde{\Phi}_n/\partial\eta$ is an even function of η . In contrast, the equation for \dot{V}_z has a change of sign on crossing the disc:

$$\dot{V}_z = -\frac{1}{\sqrt{a^2 - R_c^2}} \left. \frac{\partial\tilde{\Phi}_n}{\partial\xi} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0}, \quad \text{for } z = 0^+, \quad (26a)$$

$$\dot{V}_z = \frac{1}{\sqrt{a^2 - R_c^2}} \left. \frac{\partial\tilde{\Phi}_n}{\partial\xi} \right|_{\eta=-\sqrt{1-R_c^2/a^2}}^{\xi=0}, \quad \text{for } z = 0^-. \quad (26b)$$

However, since $\tilde{\Phi}_n$ has symmetry of reflection with respect to the plane $z = 0$, its z -derivative must vanish exactly in the equatorial plane and we have

$$\dot{V}_z = 0, \quad \text{for } z = 0, \quad (27)$$

ensuring the existence of circular orbits inside the disc. Now, according to (25), one can verify that equation (16), for inner equilibrium points, can be cast as

$$\ell_c^2 = GM_p R_c - \frac{R_c^4}{a\sqrt{a^2 - R_c^2}} \left. \frac{\partial\tilde{\Phi}_n}{\partial\eta} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} \quad \text{for } R_c < a. \quad (28)$$

On the other hand, for outer circular orbits (equilibrium points outside the disc) equation (16) becomes

$$\ell_c^2 = GM_p R_c + \frac{R_c^4}{a\sqrt{R_c^2 - a^2}} \left. \frac{\partial\tilde{\Phi}_n}{\partial\xi} \right|_{\xi=\sqrt{R_c^2/a^2-1}}^{\eta=0} \quad \text{for } R_c > a. \quad (29)$$

Equations (28) and (29) are relevant in the derivation of the quadratic epicyclic and vertical frequencies, in order to provide a criterion for the linear stability of the structures studied here.

4 LINEAR STABILITY OF STRUCTURES

As it was mentioned above, we assume the simplified model that the structures are built from particles moving in concentric circles. In this first approximation, the stability analysis of circular orbits associated to test particles provides a stability criterion for the structure, assumed to be a rotating ring of fluid (Lord Rayleigh 1916; Landau & Lifshitz 1987; Letelier 2007). For this reason, we now examine the behaviour of the epicyclic and vertical frequencies associated to quasi-circular orbits. These quantities describe the response of test particles to radial and vertical (z -direction) perturbations, when describing a circular motion. The epicycle frequency κ and the vertical frequency ν can be calculated from the effective potential Φ_n^* through the following relations (Binney & Tremaine 2008):

$$\kappa^2 = \left(\frac{\partial^2 \Phi_n^*}{\partial R^2} \right)_{(R_c, 0)}, \quad \nu^2 = \left(\frac{\partial^2 \Phi_n^*}{\partial z^2} \right)_{(R_c, 0)}. \quad (30)$$

If relation (16) is introduced in the second derivatives of Φ_n^* , we can obtain κ^2 and ν^2 as functions of R_c . Thus, values of R_c such that $\kappa^2 > 0$ and (or) $\nu^2 > 0$ corresponds to stable quasi-circular orbits under small radial and (or) vertical perturbations, respectively. Otherwise we find unstable circular orbits. Since the equatorial plane is composed of two regions, i.e. inside and outside the disc, we must define each of these quantities as piecewise functions. The quadratic epicyclic frequency for equilibrium points in the inner

zone can be written as

$$\begin{aligned} \kappa^2 = & \frac{GM_p}{R_c^3} + \frac{3R_c^2 - 4a^2}{a(a^2 - R_c^2)^{3/2}} \left. \frac{\partial\tilde{\Phi}_n}{\partial\eta} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} \\ & + \frac{R_c^2}{a^2(a^2 - R_c^2)} \left. \frac{\partial^2\tilde{\Phi}_n}{\partial\eta^2} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} \quad \text{for } R_c < a, \end{aligned} \quad (31)$$

whereas that for equilibrium points in the outer zone we have

$$\begin{aligned} \kappa^2 = & \frac{GM_p}{R_c^3} + \frac{3R_c^2 - 4a^2}{a(R_c^2 - a^2)^{3/2}} \left. \frac{\partial\tilde{\Phi}_n}{\partial\xi} \right|_{\xi=\sqrt{R_c^2/a^2-1}}^{\eta=0} \\ & + \frac{R_c^2}{a^2(R_c^2 - a^2)} \left. \frac{\partial^2\tilde{\Phi}_n}{\partial\xi^2} \right|_{\xi=\sqrt{R_c^2/a^2-1}}^{\eta=0} \quad \text{for } R_c > a. \end{aligned} \quad (32)$$

Similar expressions can be derived for quadratic vertical frequency, which inside the disc takes the form

$$\begin{aligned} \nu^2 = & \frac{GM_p}{R_c^3} + \frac{R_c^2}{a(a^2 - R_c^2)^{3/2}} \left. \frac{\partial\tilde{\Phi}_n}{\partial\eta} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} \\ & + \frac{1}{a^2 - R_c^2} \left. \frac{\partial^2\tilde{\Phi}_n}{\partial\xi^2} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} \quad \text{for } R_c < a, \end{aligned} \quad (33)$$

and outside the disc,

$$\begin{aligned} \nu^2 = & \frac{GM_p}{R_c^3} + \frac{R_c^2}{a(R_c^2 - a^2)^{3/2}} \left. \frac{\partial\tilde{\Phi}_n}{\partial\xi} \right|_{\xi=\sqrt{R_c^2/a^2-1}}^{\eta=0} \\ & + \frac{1}{R_c^2 - a^2} \left. \frac{\partial^2\tilde{\Phi}_n}{\partial\eta^2} \right|_{\xi=\sqrt{R_c^2/a^2-1}}^{\eta=0} \quad \text{for } R_c > a. \end{aligned} \quad (34)$$

Equations (31) and (33) are used to determine the conditions to be satisfied by the parameters of the system, so that the stability condition is fulfilled (relations 32 and 34 are studied in the next section). By keeping Σ_c and a constants, we can search the set of M_p values that make stable configurations. From here on we shall use $G = \Sigma_c = a = 1$ without loss of generality. In Fig. 2, at the left-hand side, we can see the behaviour of κ^2 (inside the disc) for different values of the planet's mass, in the case $n = 1$. Since it is

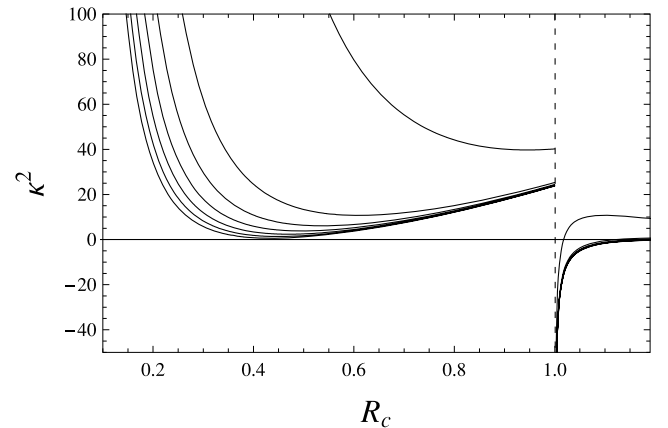


Figure 2. Behaviour of quadratic epicyclic frequency for the model with $n = 1$, for $\tilde{M}_1/M_p = 0.05, 0.4, 0.8, 1.2, 1.6, 2, 2.8$. For values larger than $\tilde{M}_1/M_p = 2.8$ (bottom curve on the left- and right-hand sides), κ^2 is negative in the equatorial plane. For smaller values, κ^2 is greater and the gap of discontinuity at $R_c = a = 1$ is smaller.

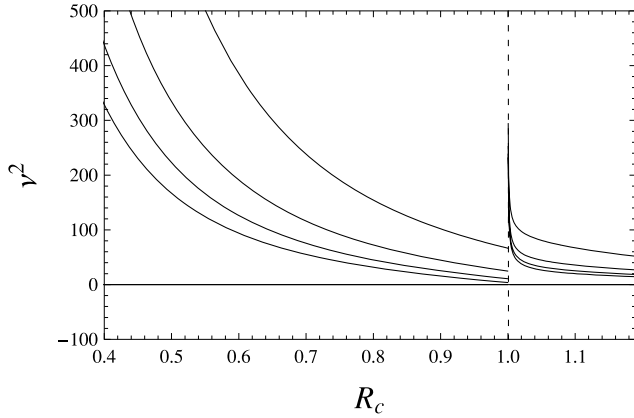


Figure 3. Behaviour of quadratic vertical frequency for the model with $n = 1$, for $\tilde{M}_1/M_p = 10^{-9}, 0.01, 0.02, 0.0485$. For values larger than $\tilde{M}_1/M_p = 0.0485$ (bottom curve on the left-hand side and top curve on the right-hand side), ν^2 is negative in the equatorial plane. For smaller values, ν^2 is greater and the gap of discontinuity is smaller.

a positive concave function with a critical point in the range $0 \leq R_c \leq a$, we can find the minimum value of M_p (or the maximum rate \tilde{M}_1/M_p) for which $\kappa^2(M_p, R_c)$ is positive in this range. We find such value by solving the simultaneous equations

$$\frac{\partial \kappa^2}{\partial R_c}(M_p^*, R_c^*) = 0, \quad \kappa^2(M_p^*, R_c^*) = 0,$$

for the variables M_p^* and R_c^* , representing the minimum value of the planet's mass and the corresponding critical radius, respectively. We can see that by increasing M_p , starting from M_p^* , we obtain increasing values for κ^2 .

Fig. 3 shows the behaviour of the quadratic vertical frequency, also for the model with $n = 1$, and we note that it is a monotonically decreasing function, in the range $0 \leq R_c \leq a$, with a minimum at $R_c = a$. In this case, to find the minimum value of M_p (which we shall denote as M_p^{**}) for which $\nu^2(M_p, R_c)$ is positive, it is enough to solve the equation

$$\nu^2(M_p^{**}, a) = 0$$

for the variable M_p^{**} . As in the previous case, we note that by increasing M_p , starting from M_p^{**} , the values of ν^2 increase.

Another feature showed in Figs 2 and 3 is that, for large values of \tilde{M}_1/M_p , the behaviour of κ^2 and ν^2 in the range $R_c > a$ is very different from the behaviour in $0 \leq R_c \leq a$. In general, we see that

$$\lim_{R_c \rightarrow a^-} \kappa^2 \neq \lim_{R_c \rightarrow a^+} \kappa^2$$

(both are finite values), but the difference between these two limits is attenuated by decreasing the ratio \tilde{M}_1/M_p . We say that there is a *gap of discontinuity* at $R_c = a$. The same statements hold for ν^2 and models with $n \geq 2$. It is clear that the gap disappears as $\tilde{M}_1/M_p \rightarrow 0$.

The behaviour of epicyclic and vertical frequencies sketched above is the same for the other models with $n = 2, 3, \dots$, so that we can compute the maximum values of the rate \tilde{M}_n/M_p leading to $\kappa^2 > 0$ or $\nu^2 > 0$. Such values are listed in Table 2, for the first 10 members of the family (11). Since in all cases $\tilde{M}_n/M_p^{**} < \tilde{M}_n/M_p^*$, we can establish that M_p^{**} is the parameter that determines the boundary between stability and instability in configurations characterized by gravitational potentials of the form (11).

It is important to note that, in these models with fixed exterior radius a , the smaller the size of the n th ring model, the larger is M_p^{**} .

Table 2. Ratios \tilde{M}_n/M_p^* and \tilde{M}_n/M_p^{**} for the first 10 members of the family of configurations represented by (11). Here M_p^* and M_p^{**} are the minimum values of the central body's mass such that $\kappa^2 > 0$ and $\nu^2 > 0$, respectively. In all cases $\tilde{M}_n/M_p^* > \tilde{M}_n/M_p^{**}$.

n	\tilde{M}_n/M_p^*	\tilde{M}_n/M_p^{**}	n	\tilde{M}_n/M_p^*	\tilde{M}_n/M_p^{**}
1	2.831 02	0.048 50	6	0.131 48	0.003 95
2	0.862 97	0.021 41	7	0.100 24	0.003 03
3	0.433 98	0.012 05	8	0.079 09	0.002 40
4	0.265 54	0.007 72	9	0.064 07	0.001 94
5	0.180 68	0.005 37	10	0.052 99	0.001 61

Stability of the Monopole–Ring System

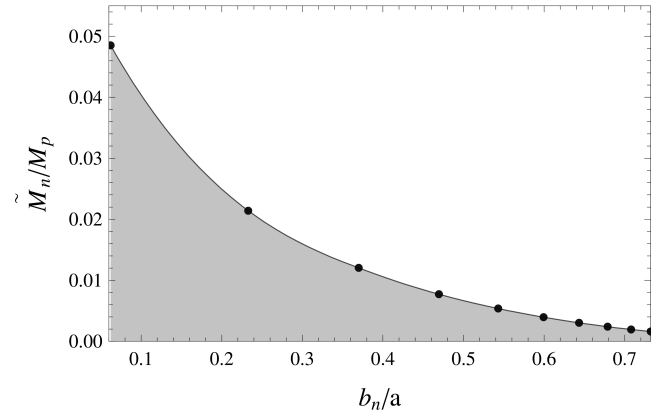


Figure 4. Correlation between the rate \tilde{M}_n/M_p^{**} and b_n/a according to the behaviour of quadratic vertical frequency ν^2 and quadratic epicyclic frequency κ^2 . The interpolation line among points is a separatrix between a stability grey region, where $\nu^2 > 0$ and $\kappa^2 > 0$, and an instability white region, where $\nu^2 < 0$ (and κ^2 may be positive or negative). The points plotted here correspond to the first 10 models, from left- to right-hand side.

The reason is that, for higher members of the family (11), the ring's mass concentration tends to be located near the outer edge and, therefore, away from the central monopole. Thus, an increasingly central body mass will be required to provide stability to the ring structure, as n grows. This fact can be glimpsed through a comparison between Tables 1 and 2, from which one might infer that there is some kind of correlation between \tilde{M}_n/M_p^{**} and the ring's size of the family of models. In Fig. 4 we show this correlation by plotting the values of Tables 1 and 2 and interpolating the corresponding points, for the first 10 members of the family. Thus we can sketch a boundary between stability and instability of self-gravitating configurations under study. In Fig. 4, points located in the grey zone correspond to parameters leading to stable configurations, whereas the points in the white region are associated to unstable configurations. We find that the points belonging to the separatrix of Fig. 4 can be fitted by the relation

$$\tilde{M}_n/M_p^{**} \approx 0.065 \exp(-4.67b_n/a). \quad (35)$$

5 THE PHASE-SPACE STRUCTURE

In this section, we shall make a description of three-dimensional motion through the study of phase-space structure associated to orbits of test particles. We are principally interested in the influence of the mass ratio in the regularity of three-dimensional bounded

orbits. The influence of the size of the ring can be inferred from it, because in the models studied here, b_n is automatically determined by \tilde{M}_n . This influence has already been analysed in relation with the stability of equatorial circular orbits and now we extend such study to the case of more general orbits (remember that the motion restricted to the equatorial plane, which is completely integrable, can be exclusively classified as stable or unstable). In order to show how the nature of bounded motion is conditioned by the linear stability of the self-gravitating structures, we will use parameters close to the critical values we have defined previously (Table 2). The influence of the quadrupolar deformation is considered in Section 6.

We shall use cylindrical coordinates and plot $z = 0$ surfaces of the section corresponding to the equations of motion (12a)–(12b) for different values of the conserved quantities E and ℓ . It is worth pointing out that we need to make explicit distinction between the orbits that cross the $z = 0$ plane at $0 \leq R \leq a$ and the ones that cross it only at $R > a$. The former are called *disc-crossing orbits* (DCO) and the latter we shall denote as non-disc-crossing orbits (NDCO). The reason is that, for each of the above situations, the origin of irregular motion is different. For the case of DCO the discontinuity in the z -component of the gravitational field (equations 26a–26b) can produce a fairly abrupt change in the curvature, leading to irregular motion (Saa 1999, 2000; Hunter 2005; Ramos-Caro, López-Suspez & González 2008). Somehow, this is a problem analogous to the case of the chaotic behaviour of Chua's circuit (Matsumoto, Chua & Komuro 1985), which is described by an autonomous system of the type $\dot{x} = f(x)$, where $f(x)$ is a piecewise function of class C^0 (continuous but not differentiable). It is the first example where the existence of such class of function leads to the existence of a chaotic attractor in a dynamical system (Madan 1993).

In contrast, in the case of other three-dimensional orbits, chaotic motion is due to the existence of saddle points in the effective gravitational potential. Such saddle points, in the particular case of a potential Φ_n^* , will be located at equatorial plane, outside the disc (note that it is not possible to define saddle points inside the disc, due to the discontinuity in the potential's z -derivative).

Equations (32) and (34) help us to investigate the existence of saddle points in the potential Φ_n^* , through the evaluation of the quantity $\Delta = (\partial^2 \Phi_n^* / \partial R^2)(\partial^2 \Phi_n^* / \partial z^2) - (\partial^2 \Phi_n^* / \partial R \partial z)^2$, which is negative when evaluated at saddle points. Since Φ_n^* is symmetric with respect to $z = 0$, the term $\partial^2 \Phi_n^* / \partial R \partial z$ vanishes at equatorial plane and the condition for existence of saddle points is reduced to $\Delta = \kappa^2 v^2 < 0$. In the right-hand side of Figs 2 and 3 we can observe the behaviour of κ^2 and $v^2 < 0$, respectively, and the product between them is showed in Fig. 5, for the case of the model with $m = 1$. For different values of \tilde{M}_1/M_p we note that there is a region of saddle points close to the outer edge, even for the maximum value 0.0485 leading to a stable configuration. When the ratio \tilde{M}_1/M_p decreases and we have even more stable configurations, the range of saddle points decreases as well as the gap between the values of $\kappa^2 v^2$ near the outer edge. This suggests that more and more stable configurations produce less and less irregular orbits.

In order to illustrate the above statements, we plot the surfaces of the section corresponding to motion around the first two models by choosing several values for the parameter M_p . We solve the equations of motion (12a)–(12b) by the Runge–Kutta fourth method with variable time-step and incorporating the Hénon algorithm (Hénon 1982), in the Laboratório de Computação Paralela (LCP) at IMECC. In the algorithm we take into account explicitly the discontinuity in the field force by implementing the set of equations (26a)–(27) as well as the piecewise transformation (17)–(20). We choose initial conditions at $z = 10^{-15}$, $V_R = 0$ and several values for R [the

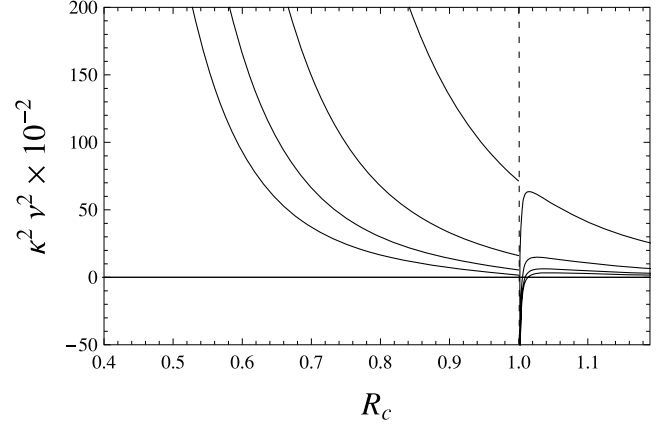


Figure 5. Behaviour of the product $\kappa^2 v^2$ for the values $\tilde{M}_1/M_p = 10^{-9}, 0.01, 0.02, 0.0485$ (curves from top to bottom). For $\tilde{M}_1/M_p = 0.0485$, the maximum value leading to a stable configuration, there is a small region of saddle points near the outer edge (right-hand side of the dashed line). This region becomes smaller as \tilde{M}_1/M_p decreases.

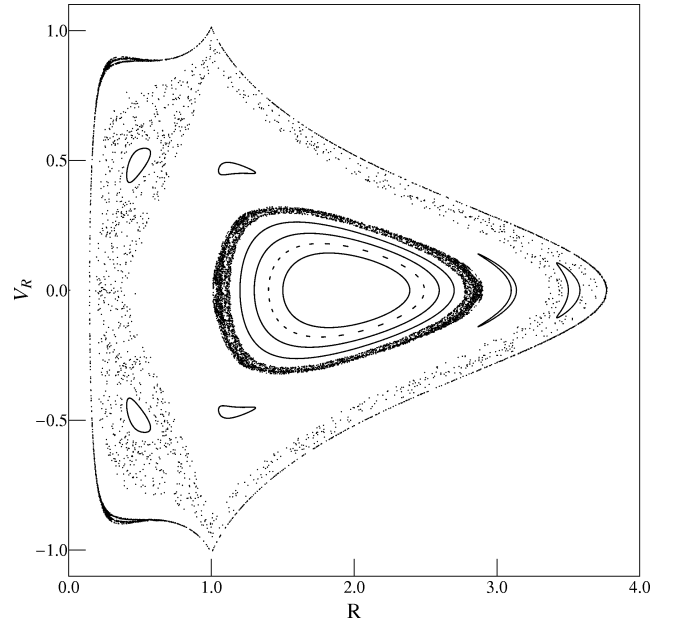


Figure 6. Surface of section for some orbits around the model with $n = 1$, with $\tilde{M}_1/M_p = 2.8$. Such a rate determines an unstable situation, where $\kappa^2 > 0$ (marginally) but $v^2 < 0$. We find two chaotic regions: (i) a prominent zone due to DCO and (ii) a smaller zone corresponding to two orbits with initial conditions next to the saddle point. In this case, we have chosen $E = -0.3$ and $\ell = 0.32$.

component V_z is given by (14) and does not vanish]. Orbits were integrated with a precision characterized by a relative error of 10^{-8} or less, in the energy conservation. Integrations were carried out on times of the order of 10^5 .

In Fig. 6 we have set $\tilde{M}_1/M_p = 2.8$, which determine a radially stable configuration, but vertically unstable. There is a variety of central KAM curves and small resonant islands outside the disc zone as well as in the inner region. They alternate with two chaotic regions, one of them is due to DCO (the most prominent) and the other, more dense, is the result of two orbits near the saddle point ($R_c = 1.014$ for this case). Here we have a situation where an unstable model admits a variety of irregular orbits. If we choose

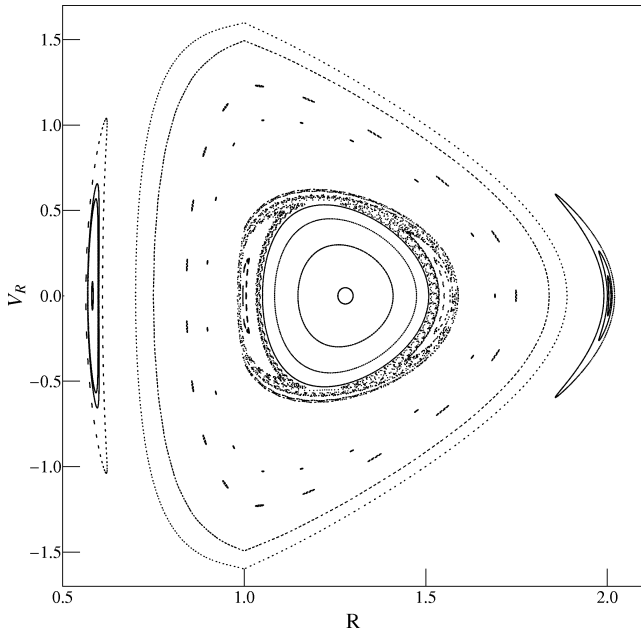


Figure 7. Surface of section corresponding to three-dimensional orbits around the model with $n = 1$, setting $\bar{M}_1/M_p = 0.0578$. The model represents an unstable situation (although very near to the separatrix of Fig. 4) and there are some chaotic orbits. The irregularity associated to these motions is due to the proximity to saddle point near the disc edge. The orbits plotted have values $E = -6$ and $\ell = 3.6$.

a more stable configuration, i.e. characterized by parameters near to the separatrix of Fig. 4, we obtain surfaces of section as shown in Fig. 7 where the chaotic region is restricted to the small central annulus enclosing the three KAM curves on the right. Note the presence of two chains of resonant islands enclosing the stochastic zone and other two small chains within it. In this case we find no chaotic motion associated to disc crossings, but only to exterior saddle points.

When we turn to the other side of the border, towards the region of stable configurations in Fig. 4, regularity is the most common feature of three-dimensional orbits. Such is the case illustrated in Fig. 8, where we find only smooth KAM curves along the surface of section, even in the case of DCO orbits. In this case we can see a chain of 29 small resonant islands enclosing another chain of three central islands. This transition from chaos to regularity, characterized by the apparition of increasing number of islands when we pass from unstable to stable structures, can be seen in Figs 9 and 10, which correspond to the model with $n = 2$. In the former there is a sequence of three prominent central islands, one of them enclosing a small chaotic region with a chain of 17 small resonant islands. In Fig. 10 the stochastic region is absent and we see only a dashed KAM curve enclosing three large islands.

It is clear that we expect a similar behaviour for the remaining members of the infinite family: the increment of the mass of the spherical central body favours the existence of isolated islands in the phase space.

6 ADDITION OF QUADRUPOLEAR TERM TO THE CENTRAL BODY

It is known that in the universe there exist many centres of attraction with a certain deviation from spherical symmetry, so we are

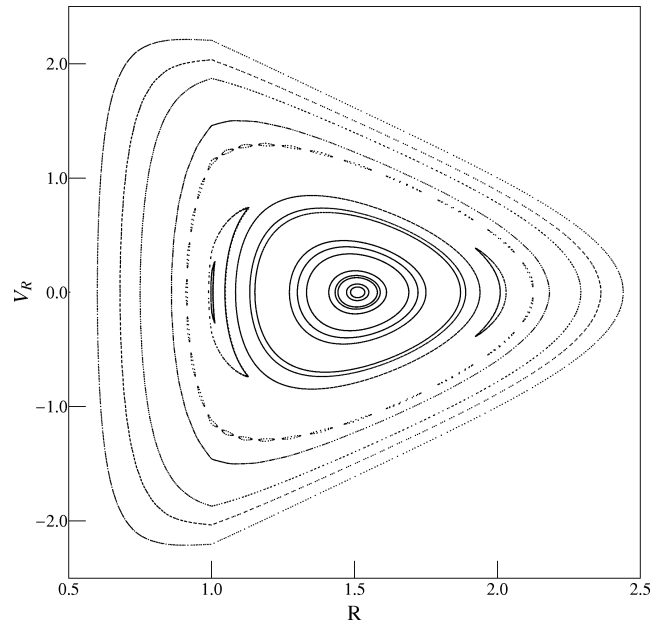


Figure 8. By maintaining the same values for E and ℓ as in Fig. 7, but decreasing \bar{M}_1/M_p to 0.0485, we find a situation when the $n = 1$ model is stable (also very near to the separatrix of Fig. 4) and we note only regular orbits.

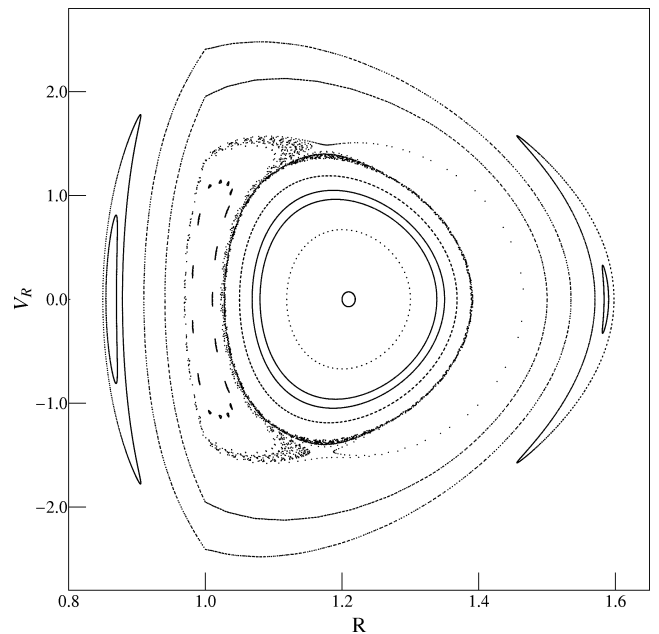


Figure 9. Surface of section corresponding to three-dimensional motion around the $n = 2$ model with $\bar{M}_2/M_p = 0.0340$, for orbits with $E = -6$ and $\ell = 4$. Since we are dealing with a configuration belonging to the unstable region of Fig. 4, there is a stochastic zone between a variety of regular KAM curves.

interested in including a quadrupolar term in the description of the central object that makes up the structures studied above and examine the effects of deformation (preserving the axial symmetry). Now the gravitational potential of the structure is

$$\Phi_n^{(\beta)} = \tilde{\Phi}_n - \frac{GM_p}{\sqrt{R^2 + z^2}} - \frac{\beta(2z^2 - R^2)}{2(R^2 + z^2)^{5/2}}, \quad (36)$$

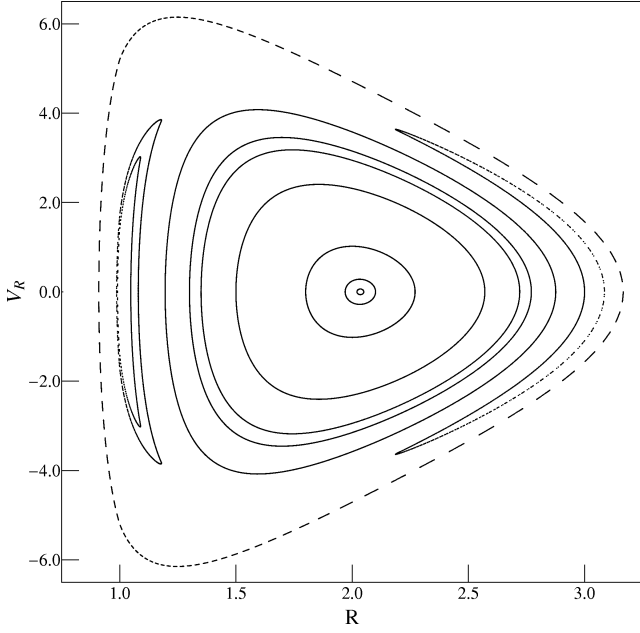


Figure 10. By maintaining the same values for E and ℓ as in Fig. 9, and choosing $\tilde{M}_2/M_p = 0.0214$, we obtain a situation when the $n = 2$ model is stable and we note only regular orbits.

where β is the quadrupolar moment that quantifies the oblate ($\beta < 0$) or prolate ($\beta > 0$) deformation of the central body. In general, it is related to the mass density $\rho(r, \theta)$ (axially symmetric) through the equation (Binney & Tremaine 2008)

$$\beta = 2\pi \int_0^\infty r^4 dr \int_0^\pi d\theta \sin\theta P_2(\cos\theta) \rho(r, \theta), \quad (37)$$

where $r = \sqrt{R^2 + z^2}$ and $\cos\theta = z/\sqrt{R^2 + z^2}$ are the spherical coordinates.

The equations of motion for test particles around such configurations are the same as equations (12a)–(12b), but replacing Φ_n by $\Phi_n^{(\beta)}$. Then, the relation for \dot{V}_R inside the ring becomes

$$\dot{V}_R = \frac{\ell^2}{R^3} + \frac{3\beta}{\eta^4} - \frac{GM_p}{R^2} + \frac{R}{a\sqrt{a^2 - R^2}} \left. \frac{\partial \tilde{\Phi}_n}{\partial \eta} \right|_{\eta=\sqrt{1-R^2/a^2}}^{\xi=0}, \quad (38a)$$

whereas the relation for \dot{V}_z , in the inner, circular orbits, remains the same as in Section 3. As a consequence, the relation that determines ℓ_c for inner circular orbits changes to

$$\ell_c^2 = GM_p R_c - \frac{3\beta}{2R_c} - \frac{R_c^4}{a\sqrt{a^2 - R_c^2}} \left. \frac{\partial \tilde{\Phi}_n}{\partial \eta} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} \quad \text{for } R_c < a, \quad (39)$$

and, for outer circular orbits,

$$\ell_c^2 = GM_p R_c - \frac{3\beta}{2R_c} + \frac{R_c^4}{a\sqrt{R_c^2 - a^2}} \left. \frac{\partial \tilde{\Phi}_n}{\partial \xi} \right|_{\xi=\sqrt{R_c^2/a^2 - 1}}^{\eta=0} \quad \text{for } R_c > a. \quad (40)$$

Therefore, the relations for the quadratic epicyclic frequency turns to

$$\kappa^2 = \frac{GM_p}{R_c^3} + \frac{6\beta}{R_c^5} + \frac{3R_c^2 - 4a^2}{a(a^2 - R_c^2)^{3/2}} \left. \frac{\partial \tilde{\Phi}_n}{\partial \eta} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} + \frac{R_c^2}{a^2(a^2 - R_c^2)} \left. \frac{\partial^2 \tilde{\Phi}_n}{\partial \eta^2} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} \quad \text{for } R_c < a \quad (41)$$

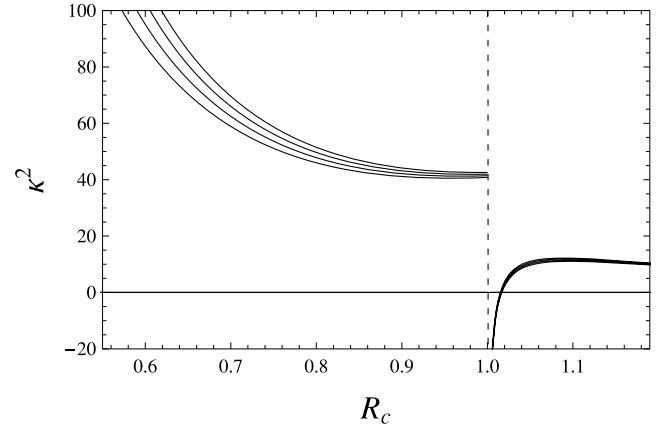


Figure 11. Behaviour of κ^2 for $\tilde{M}_1/M_p = 0.05$ and $\beta = 0.1, 0.2, 0.3, 0.4$ (curves from top to bottom). Maintaining the central monopole fixed, the epicyclic frequency grows as the quadrupolar moment decreases.

and

$$\kappa^2 = \frac{GM_p}{R_c^3} + \frac{6\beta}{R_c^5} + \frac{3R_c^2 - 4a^2}{a(R_c^2 - a^2)^{3/2}} \left. \frac{\partial \tilde{\Phi}_n}{\partial \xi} \right|_{\xi=\sqrt{R_c^2/a^2 - 1}}^{\eta=0} + \frac{R_c^2}{a^2(R_c^2 - a^2)} \left. \frac{\partial^2 \tilde{\Phi}_n}{\partial \xi^2} \right|_{\xi=\sqrt{R_c^2/a^2 - 1}}^{\eta=0} \quad \text{for } R_c > a, \quad (42)$$

whereas for the vertical frequency we now have

$$\nu^2 = \frac{GM_p}{R_c^3} - \frac{9\beta}{2R_c^5} + \frac{R_c^2}{a(a^2 - R_c^2)^{3/2}} \left. \frac{\partial \tilde{\Phi}_n}{\partial \eta} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} + \frac{1}{a^2 - R_c^2} \left. \frac{\partial^2 \tilde{\Phi}_n}{\partial \eta^2} \right|_{\eta=\sqrt{1-R_c^2/a^2}}^{\xi=0} \quad \text{for } R_c < a \quad (43)$$

and

$$\nu^2 = \frac{GM_p}{R_c^3} - \frac{9\beta}{2R_c^5} + \frac{R_c^2}{a(R_c^2 - a^2)^{3/2}} \left. \frac{\partial \tilde{\Phi}_n}{\partial \xi} \right|_{\xi=\sqrt{R_c^2/a^2 - 1}}^{\eta=0} + \frac{1}{R_c^2 - a^2} \left. \frac{\partial^2 \tilde{\Phi}_n}{\partial \xi^2} \right|_{\xi=\sqrt{R_c^2/a^2 - 1}}^{\eta=0} \quad \text{for } R_c > a. \quad (44)$$

We note that the epicyclic frequency behaves in a very similar fashion as in the previous case of a spherically symmetric central body. This can be seen in Figs 11 and 12. In the former we have chosen a certain fixed value for M_p and plotted κ^2 for some positive values of quadrupolar moment (assuming prolate deformation), whereas in the latter we have fixed β and plotted κ^2 for different values of M_p . These figures reveal that the quadratic epicyclic frequency is a positive concave function with a critical point in the range $0 \leq R_c \leq a$. Needless to clarify that it is a monotonically decreasing function if we use negative values for β . Moreover, we have to point out that this quantity has the same behaviour for the remaining models with $n = 2, 3, \dots$ (the same statement holds for the vertical frequency).

In contrast, the addition of a quadrupolar term introduces new features in the behaviour of the quadratic vertical frequency. It can be observed from Figs 13 and 14 that ν^2 is now a negative concave function with a maximum in the range $0 \leq R_c \leq a$, for positive values of quadrupolar moment (we have used the same values for the parameters as in Figs 11 and 12). It is worth clarifying that, by using negative values for β , the quadratic vertical frequency

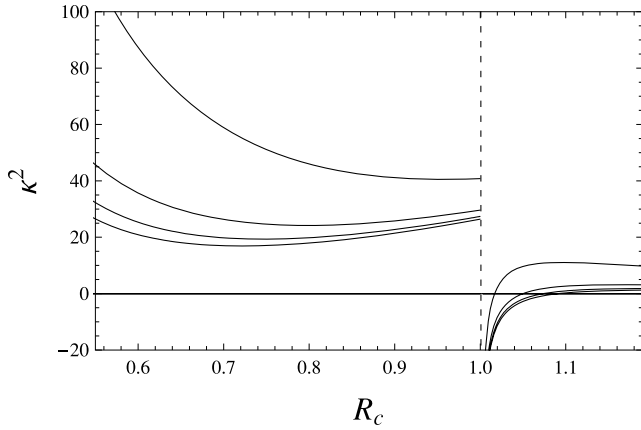


Figure 12. Behaviour of κ^2 for $\beta = 0.1$ and $\tilde{M}_1/M_p = 0.05, 0.15, 0.25, 0.35$ (curves from top to bottom). Maintaining the quadrupolar moment fixed, the epicyclic frequency grows as the central mass increases.

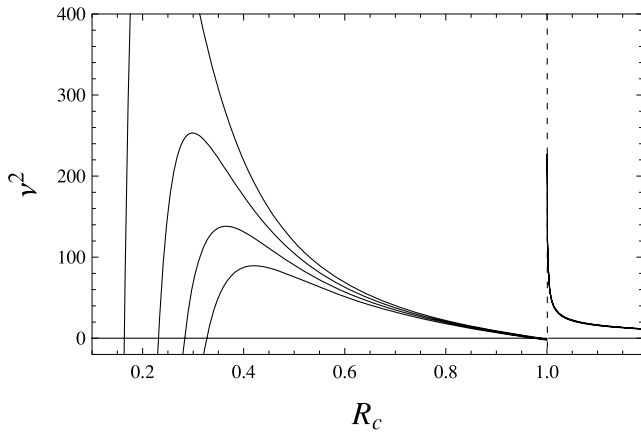


Figure 13. Behaviour of ν^2 for $\tilde{M}_1/M_p = 0.05$ and $\beta = 0.1, 0.2, 0.3, 0.4$ (curves from top to bottom). Maintaining the central monopole fixed, the range where the quadratic vertical frequency is positive grows as the quadrupolar moment decreases.

becomes a monotonically decreasing function, as in the case of a spherically symmetric central body.

According to the above statements, the task to find the limiting values of M_p and β , leading to stable configurations, is reduced to formulate the simultaneous equations

$$\nu^2(M_p^\dagger, \beta_n^\dagger, b_n) = 0, \quad \nu^2(M_p^\dagger, \beta_n^\dagger, a) = 0,$$

for the variables M_p^\dagger and β_n^\dagger , i.e. the minimum value of the central body's mass and the maximum value of the quadrupolar moment such that the quadratic vertical frequency (and evidently κ^2) is positive in the range between the cut-off radius and the outer radius. In Table 3 we have listed the corresponding values of these quantities for the first 10 members of the family. Consequently, we show the correlation between the logarithm of β_n/β^\dagger and \tilde{M}_n/M_p^\dagger , plotting the separatrix between the stability and instability (Fig. 15). Here, β_n represents the quadrupolar moment of the n th ring model and is computed using equation (37). We find that the variables $x_n = \log_{10}(|\beta_n|/\beta)$ and $y_n = \log_{10}(\tilde{M}_n/M_p^\dagger)$ can be fitted by the relation

$$y_n \approx -0.099x_n^2 + 0.345x_n - 1.592, \quad (45)$$

providing an approximate expression for the separatrix of Fig. 15.

As it was shown in Guéron & Letelier (2001), for the case of monopole–quadrupole configurations, the chaoticity is induced by

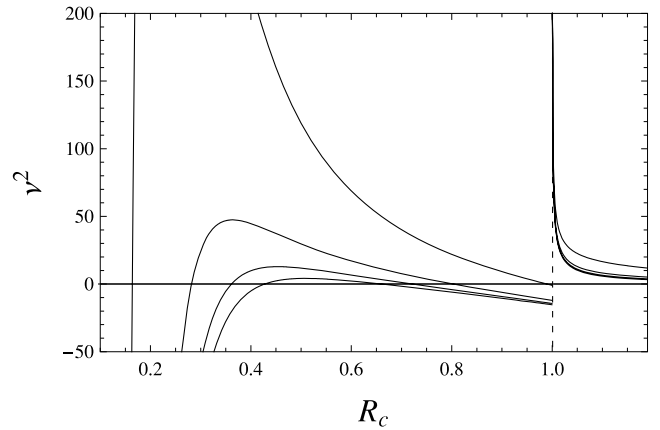


Figure 14. Behaviour of ν^2 for $\beta = 0.1$ and $\tilde{M}_1/M_p = 0.05, 0.15, 0.25, 0.35$ (curves from top to bottom). Maintaining the quadrupolar moment fixed, the range where the quadratic vertical frequency is positive grows as the central mass increases.

Table 3. Ratios \tilde{M}_n/M_p^\dagger and β_n/β^\dagger for the first 10 members of the family given by (36). Here M_p^\dagger and β^\dagger represent the minimum value of mass and maximum value of quadrupolar moment such that the structure is linearly stable.

n	β_n/β^\dagger	\tilde{M}_n/M_p^\dagger	n	β_n/β^\dagger	\tilde{M}_n/M_p^\dagger
1	-16.1076	0.048 04	6	-0.013 00	0.002 53
2	-0.559 25	0.020 24	7	-0.008 07	0.001 77
3	-0.124 08	0.010 40	8	-0.005 37	0.001 29
4	-0.047 13	0.006 02	9	-0.003 75	0.000 97
5	-0.023 01	0.003 78	10	-0.002 73	0.000 74

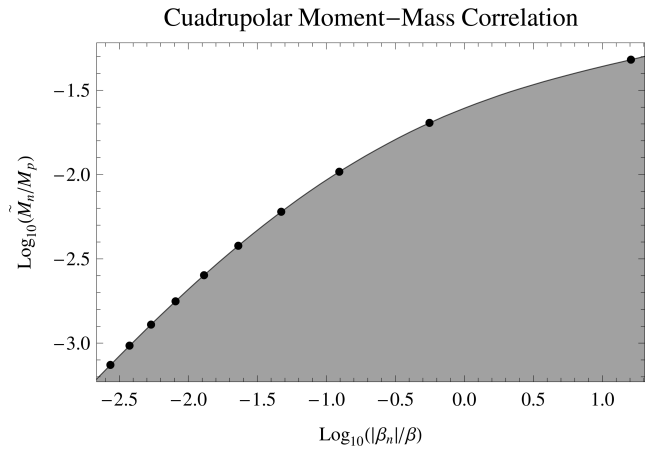


Figure 15. Correlation between the rate \tilde{M}_n/M_p and β_n/β . The grey region represents the set of values leading to stable structures, whereas the white region corresponds to the stable ones.

prolate (rather than oblate) deformation. In the case studied here, we have also a ring structure with oblate shape and one would expect two situations: (i) an ‘attenuation’ in the chaoticity, for the case of a central body with prolate deformation; (ii) only regular motions, if the central body has also oblate shape.

The surfaces of section corresponding to structures located at stability region of Fig. 15 are not very different from those shown in plots of Fig. 8 or 10. However, some differences appear when we deal with unstable configurations. For example, the outer stochastic

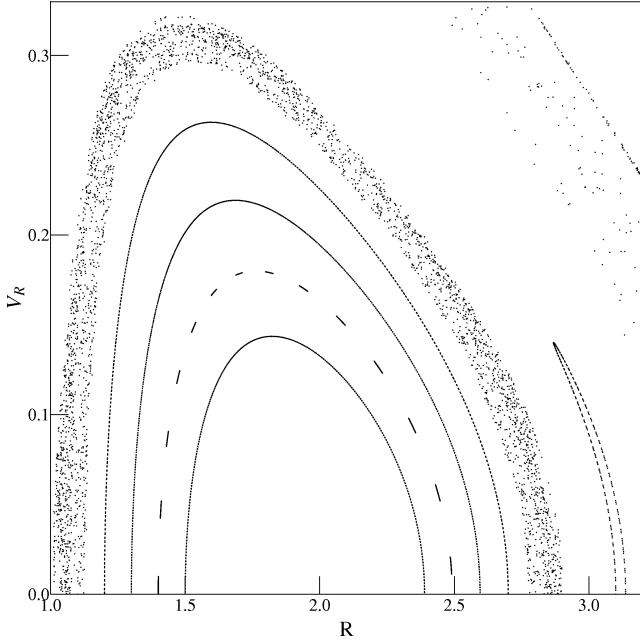


Figure 16. Detail of the central part of Fig. 6 corresponding to $n = 1$, $\bar{M}_1/M_p = 2.8$, $E = -0.3$ and $\ell = 0.32$. The inner stochastic region is generated by two orbits with initial conditions near the saddle point. The outer stochastic region is due to a DCO.

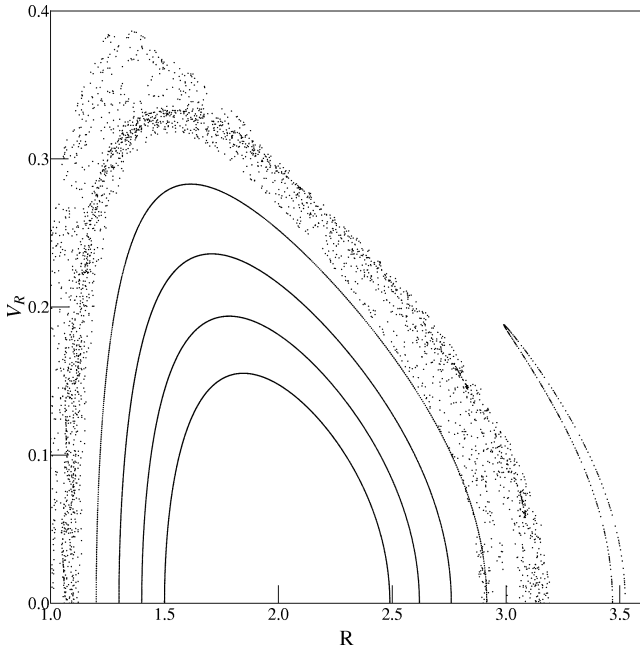


Figure 17. We have used the same parameters and initial conditions as in Fig. 16, but switching on the quadrupolar moment to $\beta = 0.03$. The stochastic regions associated to this unstable configuration are distorted, with respect to Fig. 16. The outer chaotic zone of DCO is absent, whereas the inner chaotic zone is more prominent.

region due to DCO, in Fig. 6, disappears when we turn on the quadrupolar moment, but the inner chaotic region is now most prominent and shifted slightly to the left-hand side (details are shown in Figs 16 and 17).

7 CONCLUDING REMARKS

Simple models as the ones described by relations (8) and (10) provide us an useful tool to analyse and understand the dynamics of astrophysical objects that can be modelled as a spherical (or quasi-spherical) mass surrounded by a flat-ring structure. One important and fundamental aspect in the dynamics is the stability against proper modes, if we consider in a first approximation that the ring structure is formed principally by particles moving in concentric circular orbits. It is known, in previous studies (Letelier 2007), that the structure of one ring standing alone has limited stability but, by adding a central monopole, it can improve. In this study we were able to verify such statement and find the set of values for the parameters leading to linearly stable configurations. Interestingly, we find monopole–ring configurations belonging to the stability region of Fig. 4, when dealing with parameters of the order of physical measurements performed in the Solar system. For example, a structure with the dimensions of Saturn has $R_{\min}/R_{\max} \sim 0.1$ (ratio of inner-to-outer radius) and $M_{\text{ring}}/M_{\text{planet}} \sim 10^{-8}$. Another example is a structure with $R_{\min}/R_{\max} \sim 0.4$ and $M_{\text{ring}}/M_{\text{planet}} \sim 10^{-11}$, similar to the parameters of Jupiter (Ellis, Wessen & Cuzzi 2007; Dougherty, Esposito & Krimigis 2009). We have to point out that a deeper stability analysis must include the effect of microstructure in the actual planetary rings (see e.g. Merlo & Benet 2007; Murray et al. 2008; Benet & Merlo 2009; Charnoz 2009; Charnoz, Salmon & Crida 2010).

On the other hand, there is a close connection between the stability of the configurations studied here and the regularity of three-dimensional orbits of test particles. We know that κ^2 and ν^2 are piecewise functions with very different behaviour when evaluated inside and outside the ring (Figs 2 and 3). The piece corresponding to the inner region describes the linear stability of the ring, whereas the one corresponding to the outer zone is associated to the regularity or chaoticity in three-dimensional motion (NDCO). We showed that each one of these pieces tend to be joined at the outer edge by increasing the central body mass (Fig. 5). This joining takes place in such a way that κ^2 and ν^2 tend to be positive valued functions throughout the equatorial plane. This means that more and more stable configurations lead to a gradual decrease of chaoticity in three-dimensional orbits, and the phase-space structure associated to them tends to be of the type of regular toroids.

It is worth pointing out that the most relevant contribution to the stability in the configurations studied here is provided by the mass of the central body. We can conclude that the increment of the monopolar term in the central body, rather than the decrease in the quadrupole moment, favours the existence of isolated islands in the phase space. The mass ratio and the size of the ring affect in the same way the stability of circular orbits (inside and outside the ring) and the regularity of three-dimensional motion (both DCO and NDCO). The fact that for DCO there exist a variety of regular islands throughout the phase space is relevant to confirm that the circular orbits supporting the ring structure are rather insensitive to small perturbations. Likewise, the above situation also holds for outer equatorial orbits and the associated three-dimensional motion (orbits without disc crossings) favouring the possible formation of new structures.

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